Theorem (Squeeze). Lec 5 ~ 13/03/2014

Suppose $(a_n)_{n=1}^{\infty}$, $(b_n)_{n=1}^{\infty}$, $(c_n)_{n=1}^{\infty}$ are seqs. in $\mathbb{R}$. Assume

(i) $a_n \le b_n \le c_n$.

(ii) $\lim_{n\to\infty} a_n = \lim_{n\to\infty} c_n = L$

Then $\lim_{n\to\infty} b_n = L$.

Proof. For all n, we have

$$
\begin{aligned}
|b_n - L| &= |b_n + a_n - a_n - L| \\
&\le |b_n - a_n| + |a_n - L| \\
&= (b_n - a_n) + |a_n - L| \\
&\le (c_n - a_n) + |a_n - L| \\
&= (c_n - L - a_n + L) + |a_n - L|, \\
&\le |c_n - L| + |a_n - L| + |a_n - L|.
\end{aligned}
$$

Fix $\varepsilon > 0$. There exists $N_1 \in \mathbb{N}$ s.t.

$n > N_1 \Rightarrow |a_n - L| < \boxed{\frac{\varepsilon}{3}}$

There is $N_2 \in \mathbb{N}$ s.t. $|c_n - L| < \frac{\varepsilon}{3}$ when $n > N_2$.

Take $N = \max\{N_1, N_2\}$.

If $n > N$, then $|b_n - L| < \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon$. $\square$

Example. ~~Does~~ What is $\lim_{n\to\infty} \frac{|\sin(1+n^2)|}{n^2}$?

Note that $0 \leq \frac{|\sin(1+n^2)|}{n^2} \leq \frac{1}{n^2}$.

Now, $\lim_{n\to\infty} 0 = \lim_{n\to\infty} \frac{1}{n^2} = 0$. Sq. theorem implies $\lim_{n\to\infty} \frac{|\sin(1+n^2)|}{n^2} = 0$. $\square$

Theorem. Convergent seqs. are bounded. If $(a_n)_{n=1}^{\infty}$ converges, there exists $M>0$ s.t. $|a_n| \leq M$ for all n.

Proof. Use def. of convergence with $\varepsilon = 1$. There is $N \in \mathbb{N}$ s.t.

$n \geq N \Rightarrow |a_n - L| < 1$, where $\lim_{n\to\infty} a_n = L$.

This ~~means~~ implies

$|a_n| - |L| < 1$, $|a_n| < 1 + |L|$.

Take

$M = \max\{a_1, a_2, \ldots, a_N, |L|+1\}$.

Clearly, $|a_n| \leq M$ for all n. $\square$

Theorem. Assume $\lim_{n\to\infty} a_n = a$ and $\lim_{n\to\infty} b_n = b$.

Take $\lambda \in \mathbb{R}$. Then

(i) $\lim_{n\to\infty}(a_n \pm b_n) = a \pm b$. (ii) $\lim_{n\to\infty} \lambda a_n = \lambda a$.

(iii) $\lim_{n\to\infty} a_n b_n = ab$, (iv) $\lim_{n\to\infty} \frac{a_n}{b_n} = \frac{a}{b}$.

(we assume $b_n \neq 0$ for all n, $b \neq 0$)

Proof. (i), (ii) – exercise.

Let's prove (iii). For each n, we have

$$
\begin{aligned}
|a_n b_n - ab| &= |a_n b_n + a_n b - a_n b - ab| \\
&= |a_n(b_n - b) + b(a_n - a)| \\
&\le |a_n||b_n - b| + |b||a_n - a| \\
&\le M|b_n - b| + |b||a_n - a|
\end{aligned}
$$

for some $M > 0$.

Fix $\varepsilon > 0$. There exists $N \in \mathbb{N}$ s.t.

$n \ge N \Rightarrow |a_n - a| < \frac{\varepsilon}{2(|b|+1)}$ and

$|b_n - b| < \frac{\varepsilon}{2M}$. Then

$n \ge N \Rightarrow |a_n b_n - ab| \le M \cdot \frac{\varepsilon}{2M} + |b| \frac{\varepsilon}{2(|b|+1)}$

$< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$.

(iv) Fix $\varepsilon > 0$. Consider

$\cancel{\left|\frac{a_n}{b_n} - \frac{a}{b}\right| = \left| a_n - b_n \right.}$

$$\left|\frac{1}{b_n} - \frac{1}{b}\right| = \left|\frac{b - b_n}{b b_n}\right| = \frac{|b - b_n|}{|b||b_n|}.$$

Note that $b \neq 0$.

There exists N_1 s.t. $n \geq N_1 \Rightarrow$ $|b_n - b| < \frac{|b|}{2}$. Then

$\cancel{|b_n| - |b| < \frac{|b|}{2} \text{ and } |b|}$

$$|b| - |b_n| < \frac{|b|}{2} \text{ and } |b_n| > \frac{|b|}{2}.$$

This implies

$$\left|\frac{1}{b_n} - \frac{1}{b}\right| \leq \frac{|b - b_n| 2}{|b||b|} = \frac{2|b - b_n|}{|b|^2}.$$

Because b_n converges to b, there is N s.t. $n \geq N \Rightarrow |b_n - b| < \frac{\varepsilon |b|^2}{2}$. Then $n \geq N \Rightarrow \left|\frac{1}{b_n} - \frac{1}{b}\right| < \varepsilon$. Hence $\lim_{n\to\infty} \frac{1}{b_n} = \frac{1}{b}$.

(v) follows from this and (iii). $\square$

Example. Fix $\lambda \in (0,1)$.
Set $a_n = \lambda^n$. Then $\lim_{n\to\infty} a_n = 0$.

Proof. Note $\lambda^n > 0$. Now,

$$0 < \lambda < 1 \Rightarrow 0 \cdot \lambda^n < \lambda^{n+1} < 1 \cdot \lambda^n,$$

$$0 < \underline{\lambda^{n+1} < \lambda^n}.$$ This means

λ^n is monotone decreasing.

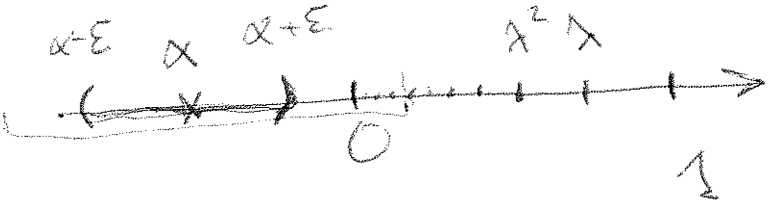
Such sequences always converge
(we prove this later).

Thus, $\lim_{n\to\infty} \lambda^n = \alpha$ for some $\alpha \in \mathbb{R}$.

Lec. 6

Claim. $\alpha = 0$.

a) Assume $\alpha < 0$.



Use def. of limit.

Take $\varepsilon = \frac{|\alpha|}{2} = -\frac{\alpha}{2}$.

There exist $N \in \mathbb{N}$ s.t. $n > N$

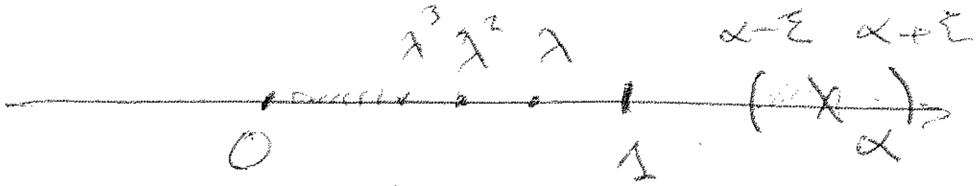
$$\Rightarrow |\lambda^n - \alpha| < -\frac{\alpha}{2}$$

$$\Rightarrow \frac{\alpha}{2} < \lambda^n - \alpha < -\frac{\alpha}{2}$$

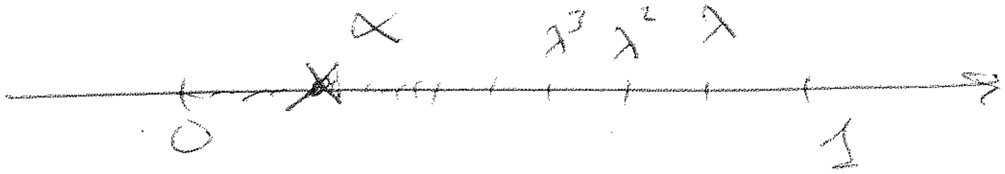
$$\Rightarrow \lambda^n < \frac{\alpha}{2} < 0.$$ Contradiction.

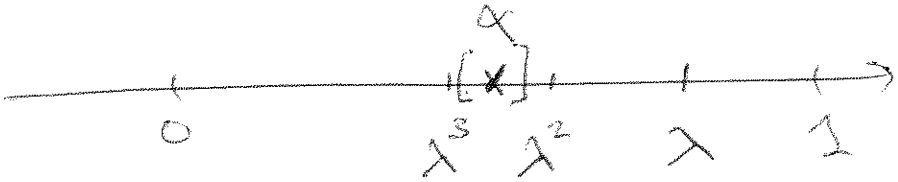
Hence $\alpha \geq 0$.

b) $\alpha \geq 1$. Same argument works.



c) What if $\alpha \in (0,1)$.





Choose $\varepsilon = \frac{\alpha}{\lambda} - \alpha$. Then there is $N \in \mathbb{N}$ s.t. $n > N \Rightarrow |\lambda^n - \alpha| < \varepsilon$.

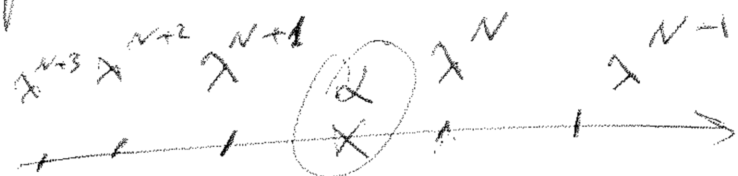
This means

$$\alpha - \varepsilon < \lambda^N < \alpha + \varepsilon$$

$$\Rightarrow \cancel{2\alpha - \frac{\alpha}{\lambda}} < \lambda^N < \frac{\alpha}{\lambda}$$

$$\Rightarrow \lambda^{N+1} < \alpha.$$

But this is impossible! Why?



Because λ^n is monotone decreasing.

Contradiction. Thus, $\alpha = 0$. $\square$

Upper and lower bounds of sets.

Def. Suppose $\Omega \subset \mathbb{F}$, where $\mathbb{F}$ is an ordered field. $b \in \mathbb{F}$ is an upper (lower) bound of Ω if

$b \geq x$ $(b \leq x)$ for all $x \in \Omega$.

b is the least upper bound (or supremum) if

(i) b is an upper bound of Ω

(ii) $b \leq c$ for every upper bound c of Ω

The greatest lower bound (or infimum) is defined analogously.

Def. We say $\mathbb{F}$ has the least upper bound property if every set which has an upper bound in $\mathbb{F}$ has a least upper bound in $\mathbb{F}$.

Examples. $\mathbb{F} = \mathbb{R}$. $\Omega = [0, 1]$.

$\sup \Omega = 1$.

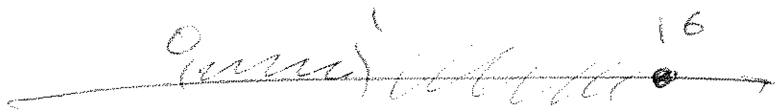
$\inf \Omega = 0$.

Note. sup and inf are unique.

Proof. Let s_1 and s_2 be least upper bounds of Ω. Then, by (i), they are both upper bounds. By (ii), $s_1 \leq s_2$ and $s_2 \leq s_1$. Thus, $s_1 = s_2$.

$\Omega = (0, 1)$, $\sup \Omega = 1$,

$\Omega = (0, 1) \cup \{16\}$, $\sup \Omega = 16$, $\inf \Omega = 0$.



$\Omega = \mathbb{N}$,
$\sup \Omega$ does not exist.
$\inf \Omega = 1$.

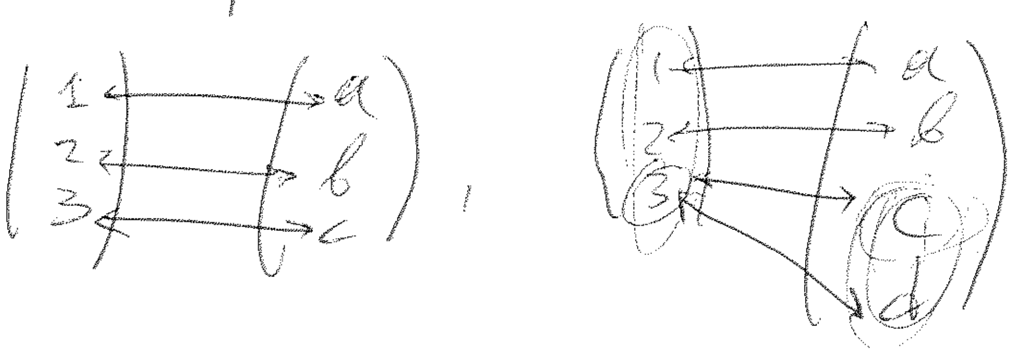
$\Omega = \mathbb{Q}$, $\sup \Omega$, $\inf \Omega$ do not exist.

If $\sup \Omega = \inf \Omega$, then Ω has only one point.

Cardinality.

We say two sets Ω_1 and Ω_2 have the same cardinality if there is a one-to-one correspondence between them.

Example.



<u>Def.</u> Ω is <u>countable</u> if it has the same cardinality as $\mathbb{N}$.

Why is $\mathbb{Z}$ contable?

$$\begin{array}{ccccccccc} -4 & -3 & -2 & -1 & 0 & 1 & 2 & 3 & 4 \\ & & & \uparrow & \uparrow & \uparrow & \uparrow & & \\ & & & 3 & 1 & 2 & 4 & & \end{array}$$

Is $\mathbb{Q}$ countable?

<u>Note:</u> an infinite subset of a countable set is countable.

$\mathbb{Q}$ is a subset of $\{(p,q) \mid p \in \mathbb{Z}, q \in \mathbb{N}\}$.

Why is the set of all such pairs countable?

$$\begin{array}{cccc} \overset{1}{\downarrow} & & & \\ (0,1) & (0,2) & (0,3) & (0,4) \\ (1,1) & (1,2) & (1,3) & (1,4) \\ (2,1) & (2,2) & (2,3) & (2,4) \end{array}$$

Exercise: do this right.

Why is $\mathbb{R}$ not countable?

~~Take~~ Assume $\mathbb{R}$ is countable.
~~Th~~ Write the real ~~numb~~ numbers of the form $0.0011010\ldots$ (infinite fractions of 0 and 1) in a sequence.

$$0.\overset{1}{\cancel{0}}0000\ldots$$
$$0.0\overset{0}{\cancel{1}}000\ldots$$
$$0.11\overset{1}{\cancel{0}}01\ldots$$
$$0.001\overset{0}{\cancel{1}}0\ldots$$

Then take
$0.1010\ldots$
This is different from all.
But we assumed we listed all.
contradiction. $\square$

<u>Ordered field with l.u.b. property.</u>